The loci of 9, 10, 11 can be found by solving cubic equations or by means of Calculus. In case 11, it may also be solved as follows:

$$x/y=-\frac{b^4u^3}{a^4v^3} \text{ or } u/v=-\frac{a}{b}\sqrt[3]{\frac{ax}{by}}. \quad \therefore u=-\frac{av}{b}\sqrt[3]{(ax/by)}, \quad v=-\frac{bu}{a}\sqrt[3]{(by/ax)},$$

from values of x, y we get $u+x=\dfrac{a^2x}{b^2}\sqrt[3]{\dfrac{b^2y^2}{a^2x^2}}$, $v+y=\dfrac{b^2y}{a^2}\sqrt[3]{\dfrac{a^2x^2}{b^2y^2}}$.

These relations also apply to the hyperbola.

ON CERTAIN PROOFS OF THE FUNDAMENTAL THEOREM OF ALGEBRA.

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I.

Until an American text-book on Higher Algebra shall appear, the great majority of American students will probably continue to approach the Fundamental Theorem of Algebra through the well known English texts of Chrystal, Burnside and Panton, or Todhunter. It is therefore to be regretted that these texts, though they aim at considerable rigor in demonstration, fail when it comes to this most important theorem. Yet it is this very theorem where rigor means all, for the mere fact which the theorem embodies, is well known to every student long before he reaches the demonstration in question. It is my pnrpose to point out as briefly as possible certain of these tacit assumptions employed by the several authors, in the hope that if it is not desirable to entirely avoid them, as has been done by Weber in his classic text on algebra, they may at least be explicitly stated as such, in future editions of these and other texts.

II.

Chrystal's* proof is in outline as follows: To prove that one value of z, in general a complex number, can always be found which causes the rational integral function, $f(z)$, to vanish, "we have to show that a value of z can always be found which shall render mod $f(z)$ smaller than any assignable quantity. This will be established if we can show that however small mod $f(z)$ be, provided it be not zero, we can always, by properly altering z, make mod $f(z)$ smaller still." The proof now consists in showing that so long as $f(z)\neq 0$, an increment h of z may be so determined that

$$\text{mod}\, f(z+h) < \text{mod}\, f(z);$$

*Chrystal, *Algebra*, I, Chapter XII, §22.

that is, so long as mod$f(z)$ is not zero, it may be diminished, and hence it is concluded that one value of z can be found such that mod$f(z)=0$, that is, such that $f(z)=0$.

The assumption here is, that if for every value of $f(z)$ an h can be found such that mod $f(z+h)<$mod $f(z)$, $f(z)$ must necessarily vanish for some value of z. This is a non sequitur. The inference is not warranted that a function which permits of diminution for every value of the argument possesses necessarily a zero value. If, for example,

$$f(x)=\left(\frac{1+x^2}{x^2}\right)^{x^2+1},$$

$f(x)$ has no zero value, yet for every value of x an h may be found such that

$$f(x+h)<f(x).$$

III.

Burnside and Panton* retain through successive editions of their work the following proof: Suppose $z=x+iy$ is represented by the point (x, y) in the z-plane and its image $w=f(x+iy)=u+iv$ by the point (u, v) in the w-plane. If a point $z_1=z+h$, sufficiently near z, is made to describe a small closed curve about z, its image $w_1=w+k$ will describe a corresponding small closed curve about w. Suppose, now, that there is no value of z which makes $f(z)=0$, then there must be some point z, whose image w is nearer the origin than that of any other point z. But by properly selecting the path of z_1, the path of its image w_1 can be made to pass between w and the origin. This is contrary to the supposition that w is the nearest possible position with reference to the origin, consequently no value different from zero can be the least possible value of mod $f(z)$.

The unwarranted assumption in this proof is, that there must be some z whose image w is necessarily nearer the origin than that of any other z. Nor is the objection removed by letting w be the image nearer the origin (or at least as near as) the image of any other point z, as has been done by Fine† in his otherwise admirable proof of this theorem. The real objection is that w is assumed to possess a nearest position to 0 at all, that is, that the function mod $f(z)$ is assumed to possess a minimum, a supposition the contrary of which is certainly conceivable and may be legitimately held so long as nothing to the contrary has been established. In fact, it is possible to construct an algebra in which the assumption does not hold. We need only to confine ourselves to the domain of real numbers which are greater than unity. Let x be any number of this domain, then

$$f(x)=\frac{x^3+2x}{1+2x^2}$$

*Burnside and Panton, *Theory of Equations*, Fourth Edition, Vol. I, §123.
†Fine, *The Number-system of Algebra*, §54.

has no minimum value in this domain. For so long as x is real and less than unity, $f(x)$ also is real and less than unity, and

$$f(x) < x,$$

that is, for every conceivable value of $f(x)$ of our domain we can construct another which is less.

IV.

Todhunter,* in spite of his extreme caution in dealing with demonstrations of a critical nature, also assumes the existence of a minimum without recognizing it. He himself outlines his argument in the words, "Since U^2+V^2 (that is, mod $f(z)$) is always a real positive quantity, if it can not be zero there must be some value which can not be diminished; but we shall now prove that if U^2+V^2 have any value different from zero we can diminish that value by a suitable change in the expression which is substituted for x, so that it follows that U^2+V^2 must be capable of the value zero, that is, U and V must vanish simultaneously." The argument as here stated really involves two unwarranted assumptions:

1. That the function U^2+V^2 necessarily possesses a minimum,
2. That a function which permits of being indefinitely diminished must necessarily approach the limit zero.

The second of these assumptions is removed in a postscript article, but the first is ignored.

It may be mentioned in conclusion that the assumption of the existence of a minimum occurs in the common source of the various proofs which I have cited, the so-called first proof of Cauchy,† or going back still further in Legendre‡ who first developed the essential principles of Cauchy's proof.

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A GENERAL NOTATION FOR VECTOR ANALYSIS.

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Vector analysis is now a little over half a century old. As compared with most other branches of mathematical thought its cultivation is very recent. As we look back over the history of the development of the mathematical notations we see that at first there was great diversity in the signs used, but that this diversity in time gave place to uniformity.

*Todhunter, *Theory of Equations*, Chapter II.

†A. Cauchy, *Course d'Analyse*, Chapter XII, §1, Theorem I.

‡Legendre, *Theorie des Numbers*, 1re Part, §XIV; In the German translation by Maser, §119.